Ageing in spin-glasses in three, four and infinite dimensions

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Abstract
The spin update engine (SUE) machine is used to extend, by a factor of 1000, the
timescale of previous studies of the ageing, out-of-equilibrium dynamics of the
Edwards–Anderson model with binary couplings, on large lattices ($L = 60$).
The correlation function, $C(t + t_w, t_w)$, t_w being the time elapsed under a
quench from high-temperature, follows nicely a slightly-modified power law
for $t > t_w$. Very small (logarithmic), yet clearly detectable deviations from the
full-ageing t/t_w scaling can be observed. Furthermore, the $t < t_w$ data show
clear indications of the presence of more than one time sector in the ageing
dynamics. Similar results are found in four dimensions, but a rather different
behaviour is obtained in the infinite-dimensional $z = 6$ Viana–Bray model.
Most surprisingly, our results in infinite dimensions seem incompatible with
dynamical ultrametricity. A detailed study of the link correlation function is
presented, suggesting that its ageing properties are the same as for the spin
correlation function.

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1. Introduction

Spin-glasses [1–4] were discovered to *age* even on human timescales some 20 years ago [5].
Ageing is nicely demonstrated, for instance, in measures of the thermoremanent magnetization
(see, e.g., [6]): in the presence of a magnetic field, cool a spin-glass from room temperature

to the working temperature, T, below its glass temperature; hold the magnetic field for a while (the time elapsed will be called t_w hereafter), then switch off the field and record the time-decay of the magnetization $M_{t_w}(t)$. Not only is this decay very slow but also, even for the longest t_w tried up to now, $M_{t_w}(t)$ strongly depends on t_w (larger the t_w, slower the $M_{t_w}(t)$ decays). It has slowly become clear that the important information obtained from experiments in spin-glasses regards dynamic out-of-equilibrium effects, such as this one or the more sophisticated memory and rejuvenation effects [7, 6]. Although there has been a burst of theoretical activity in out-of-equilibrium dynamics [8–10], it is still not clear[5] how out-of-equilibrium experiments will help us choose among the conflicting theoretical views on the nature of spin-glasses: replica symmetry breaking [3] (RSB), the droplets picture [14] and the intermediate TNT picture [15]. The situation is further complicated by the fact that detailed theoretical predictions (to be confronted with experiments) can be extracted from models only through Monte Carlo simulations [9]. It is worth recalling that numerical results on out-of-equilibrium dynamics have cast some doubts even on the usefulness of the Edwards–Anderson model for describing physical spin-glasses [16] (see, however, [17] for some reassuring results).

It is thus clear that one needs to address quantitatively the time-decay of $M_{t_w}(t)$ or equivalently, given the fluctuation–dissipation theorem [8], the time-dependent correlation function[6] in the absence of a magnetic field:

$$C(t, t_w) = \frac{1}{N} \sum_i \langle \sigma_i(t + t_w)\sigma_i(t) \rangle. \tag{1}$$

Now, it seems to be a fact of general validity in out-of-equilibrium dynamics [8] that $C(t, t_w)$ behaves differently in different *time sectors*. Losing generality[7] for the sake of clarity, this amounts to saying that it can be decomposed as

$$C(t, t_w) = \sum_i f_i \left(\frac{(t + t_w)^{1-\mu_i} - t_w^{1-\mu_i}}{1 - \mu} \right). \tag{2}$$

Here, f_i are smooth, decreasing functions that tend to zero at infinity, and such that $f_i(0)$ is of order one. It follows that if $t \ll t_w^{\mu_i}$ the ith time sector contributes the constant value $f_i(0)$ to $C(t, t_w)$, while for $t \gg t_w^{\mu_i}$ it contributes nothing. In other words, the ith time sector is active only for $t \sim t_w^{\mu_i}$ (note that the different time sectors get neatly separated only in the limit of very large t_w). Not much is known about the exponents μ_i defining the different time sectors. With this popular parametrization [8, 6], one has $0 \leqslant \mu_i \leqslant 1$. For the simple case of the coarsening dynamics (domain growth) of an ordered ferromagnet [18], only two time sectors are needed for a complete description: $\mu_1 = 0$ describing the stationary, t_w independent dynamics found at small t, and $\mu_2 = 1$ describing the *full-ageing* situation where the correlation function depends on the ratio t/t_w. Also the spin-glass dynamics has been experimentally claimed [6] to be ruled only by two time sectors: $\mu_1 = 0$ and $\mu_2 = 0.97$. The second time sector is slightly but clearly different from the full-ageing t/t_w behaviour ($\mu = 1$) and is thus named sub-ageing. However, a very recent experiment [19] seems to indicate that the sub-ageing behaviour is just an artefact of the finite time needed to cool the system down to the working temperature (a limitation not suffered in numerical simulations). Using their fastest cooling protocol, Rodriguez *et al* [19] have found a clear full-ageing behaviour $\mu_2 = 0.999$. Furthermore, the role of the stationary time sector ($\mu_1 = 0$) to describe the data is far less critical than previously [6] thought. At this point, we wish to make two comments:

[5] A very encouraging experiment [13] measuring the violation factor of the fluctuation–dissipation theorem [11, 12] has been recently reported.

[6] Actually, in the ageing regime they are related by a very smooth function [11, 12].

[7] The formulation presented here cannot describe logarithmic domain growth, for instance, see in [8] for the general framework.

- The presence of more than two time sectors seems to be a crucial requirement for the validity of the dynamic version of the usual ultrametric replica symmetry breaking description of spin-glasses with an infinite number of replica symmetry breakings.

- If the largest μ exponent is 1, as implied by this popular parametrization [6, 8], for the very large t_w achieved in experiments (1 s means roughly $t_w = 10^{12}$), it is quite possible that all the faster sectors have already died out since $t_w^{\mu_i} \ll t_w \sim t$. A full-ageing ansatz could well describe the data, specially if the second-largest μ exponent is significantly smaller than one. In this respect, a numerical simulation could have a better chance of observing the different time sectors.

To conclude this introduction, let us recall the main results of previous intensive numerical studies of ageing dynamics [9] (for a more recent extensive numerical study of the ageing dynamics with a different focus see [16]). Indeed, it was found [9] that the correlation function at short times ($t \ll t_w$) could be described as a power law with a temperature-dependent exponent ($C(t, t_w) \sim t^{-x(T)}$). The exponent $x(T)$ was found to be fairly small ($x(T = 0.5T_{\rm c}) = 0.015$) On the other hand, at long times a power-law decay with a different exponent, $\lambda(T, t_w)$, was observed ($C(t, t_w) = t^{-\lambda(T, t_w)}$ for $t \gg t_w$). Yet, a cross-over functional form was proposed [20] (see also [9]), implying that $\lambda(T, t_w)$ saturates to $\lambda(T)$ for large t_w:

$$C(t, t_w) = t^{-x(T)} \Phi\left(\frac{t}{t_w}\right). \tag{3}$$

The cross-over function was proposed to be decreasing, smooth and to have the asymptotic behaviour $\Phi(y) \sim$ constant for small y and $\Phi(y) \sim y^{-\lambda(T)+x(T)}$ for large y. However, in the spanned timescales [9] ($t_w \leqslant 10^5$, $t \leqslant 10^6$), $\lambda(T)$ was actually found to significantly depend on t_w. Yet, if one assumes that $\lambda(T, t_w)$ saturates to $\lambda(T)$ for large t_w, so that equation (3) could hold, one would speak of a dynamics with effectively three time sectors: $\mu_1 = 0$, $\mu_2 = 1 - x(T)/\lambda(T)$ and $\mu_3 = 1$ (to derive this, assume that $\Phi(y)$ is an analytical function, such that $(1 + y)^{-\lambda(T)+x(T)}$). However, since $x(T)/\lambda(T)$ was found [9] to be 0.1 or smaller, it would not be easy to separate the μ_2 and μ_3 sectors. On the other hand, it is clear that equation (3) describes a non-ultrametric dynamics.

In this paper, by extensive Monte Carlo simulations using the dedicated SUE machine among others [21–23], we will show that there are small yet quite significant deviations from equation (3) in finite dimensions. Surprisingly enough, our results in infinite dimensions are rather different, and suggest that dynamical ultrametricity could *not* hold in (some) statically ultrametric systems. As the reader will note, heavy use of data fitting will be made in the following. Unfortunately, there is no precise theory that tells us which should be the precise functional form of $C(t + t_w, t_w)$, so it is difficult to theoretically justify many of the fits. Also the very good power-like behaviour we found cannot be recovered from analytic computations. Here we are doing some kind of exploratory work, trying to guess what could be a reasonable form that well represents the data. This may be useful for a number of reasons: if we can do good fits at given value of t_w it is rather more convenient to look to the dependence of the fit parameter on t_w than to the data themselves (sometimes scaling plots may be misleading), good fits may evidentiate some behaviour that could be eventually analytically derived, and fits can also be used to extrapolate data to large times. Finally, note that as an outcome of this strategy, some of the fits that we are doing could also be useful in analysing experimental data.

The layout of the rest of this paper is as follows. In section 2 we describe our simulations. In section 3 we concentrate on the spin–spin correlation function, presenting a new parametrization of the function $C(t, t_w)$, and discussing the possibility of numerically studying the existence of more than two time sectors. In section 4 we focus on the ageing behaviour of the link-overlap and the link correlation function (defined in section 2). We shall

conclude that even in the limit of infinite waiting time, the link correlation function ages. Finally, we present our conclusions in section 5.

2. The simulation

We have studied the three-dimensional Ising spin-glass defined on a cubic lattice ($L \times L \times L$) with helicoidal boundary conditions[8]. The Hamiltonian is

$$\mathcal{H} = - \sum_{<i,j>} \sigma_i J_{ij} \sigma_j. \tag{4}$$

The large timescale and lattice sizes simulated have been possible due to the use of a dedicated computer, SUE (spin update engine, Universidad de Zaragoza), based on programmable components, and achieving an update speed of 0.22 ns per spin. Details about the machine can be found in [21–23].

The volume of the system is $V = L^3$, σ_i are Ising variables, J_{ij} (uncorrelated quenched disorder) are ± 1 with equal probability and the sum is extended to all pairs of nearest neighbours. The choice of helicoidal boundary conditions is mandatory (for us) because the hardware of the SUE machine has been optimized for them. During the simulation we have measured the following quantities:

$$C(t, t_w) = \frac{1}{N} \sum_i \langle \sigma_i(t + t_w) \sigma_i(t) \rangle \tag{5}$$

$$C_{\text{link}}(t, t_w) = \frac{1}{DN} \sum_{\mu=1}^{D} \sum_i \langle \sigma_i(t + t_w) \sigma_{i+\mu}(t + t_w) \sigma_i(t_w) \sigma_{i+\mu}(t_w) \rangle \tag{6}$$

$$q(t_w) = \left| \frac{1}{N} \sum_i \langle \sigma_i^{(1)}(t_w) \sigma_i^{(2)}(t_w) \rangle \right| \tag{7}$$

$$q_{\text{link}}(t_w) = \frac{1}{DN} \sum_{\mu=1}^{D} \sum_i \langle \sigma_i^{(1)}(t_w) \sigma_{i+\mu}^{(1)}(t_w) \sigma_i^{(2)}(t_w) \sigma_{i+\mu}^{(2)}(t_w) \rangle \tag{8}$$

in the above equations, D is the spatial dimension, $i + \mu$ stands for the nearest neighbour in the μ direction, while the superscripts (1) and (2) refer to the replica index (real replicas: pair of systems evolving independently with the same couplings $J_{i,j}$). The notation $\langle \cdots \rangle$ refers both to average over thermal histories (random numbers) and over disorder realizations. Let us just recall that since the starting configurations are random, we have explored the so-called $q = 0$ sector [3], in which one expects dynamical correlation functions to be self-averaging.

Given the unique features of the SUE machine we have preferred to use it for very long runs in a rather small number of samples. The lattice sizes studied have been $L = 20$, $L = 30$ and $L = 60$. We have considered three values of the inverse temperature: $\beta = 1.25$, 1.67 and 2.0 (hereafter to be referred to as $T = 0.8$, 0.6 and 0.5, respectively). The critical temperature for this model is $T_c = 1.14(1)$, thus the selected temperatures are $0.7T_c$, $0.53T_c$ and $0.44T_c$, respectively. Given the very slow growth of the spin-glass coherence length (see, e.g., [9, 24]) one should not expect notable finite-size effects even for the $L = 20$ lattice. However, as is well known, sample-to-sample fluctuations in $C(t, t_w)$ decrease fast with growing system size.

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[8] Let (x, y, z) be the lattice coordinates of spin number $i \equiv x + yL + zL^2$, then the coordinates of the three nearest neighbours (in the three positive directions) are given by: $i_1 = (i + 1) \bmod L^3$, $i_2 = (i + L) \bmod L^3$ and $i_3 = (i + L^2) \bmod L^3$.

Table 1. SUE simulation details.

L	T	Number of iterations	Number of samples
60	0.5	10^9	16
60	0.6	8×10^8	16
60	0.8	5×10^8	16
30	0.5	5×10^8	32
30	0.8	5.5×10^8	32
20	0.5	2×10^9	32

We can sum up the details of the SUE simulations in table 1. The SUE time-step corresponds to 8192 full-lattice sequential heat-bath updates. For $C(t, t_w)$ and $C_{\text{link}}(t, t_w)$ we have selected the t and t_w values in a logarithmic scale. These values correspond actually to $8192 \times 2^{0.25i}$, $i = 1, \ldots$. The number of simulated samples was 16 for the $L = 60$ systems and 32 for the $L = 20, 30$ systems. Here, we will only present the results for $L = 60$, since the data for $L = 20$ and $L = 30$ are fully compatible with them, but far noisier. Yet, much more accurate data have been obtained for moderate t and t_w from simulations on a PC (see below) of an $L = 33$ system.

To have some data at times shorter than SUE step 8192, we have simulated (using a heat bath) on a personal computer 80 samples with $L = 60$ ($t_w \leqslant 6877, t \leqslant 65\,519$).

In addition, we have performed Metropolis simulations on personal computers for the same model in 3D ($L = 33, T = 0.8 = 0.7T_c$, $t_w \leqslant 2^{18}, t \leqslant 2^{22}$), 4D ($L = 37$, $T = 1.2 = 0.6T_c, t_w \leqslant 2^{18}, t \leqslant 2^{22}$) and in the infinite-dimensional Viana–Bray model ($z = 6, N = 5 \times 10^6, T = 0.8 = 0.34T_c, t_w \leqslant 2^{13}, t \leqslant 2^{17}$ and $N = 10^6$ $T = 0.8, t_w \leqslant 2^{19}, t \leqslant 2^{16}$). Even if the times were shorter than in SUE, the number of simulated samples is much larger. The typical statistical error in correlation functions (as calculated from the sample-to-sample fluctuations) obtained in the PC was 20 times smaller than the statistical errors of the SUE results.

3. Ageing dynamics

We have found in finite dimensions ($D = 3$ and $D = 4$) that the correlation function $C(t, t_w)$ can be nicely fitted for $t \geqslant t_w$ as

$$C(t, t_w) = A(t_w) \left(1 + \frac{t}{t_w} \right)^{-1/\alpha(t_w)} \qquad t > t_w. \tag{9}$$

As can be seen in figures 1–3, for a wide range of t/t_w and t_w and all three temperatures a simple power-law decay seems enough to describe the data, although the coefficients $A(t_w)$ and $\alpha(t_w)$ clearly depend on temperature (note that $\alpha(t_w)$ is just the inverse of the $\lambda(T, t_w)$ exponent of Rieger *et al* [9]). The behaviour in infinite dimensions ($z = 6$ Viana–Bray model) is rather different and will be discussed at the end of this section.

It is clear that the waiting-time dependence of the prefactor $A(t_w)$ and the exponent $\alpha(t_w)$ is of utmost importance. Should $A(t_w)$ and $\alpha(t_w)$ tend to constant non-vanishing values, dynamic ultrametricity [8] would not hold, implying that the usual dynamic formulation of the ultrametric approach to continuous replica symmetry breaking should be modified. Furthermore, we have been unable to find a divergence law for $\alpha(t_w)$ compatible with dynamic ultrametricity, if $A(t_w)$ has a non-vanishing limiting value for large t_w. Thus, we tend to believe that dynamical ultrametricity implies that $A(t_w)$ should vanish in the large t_w limit.
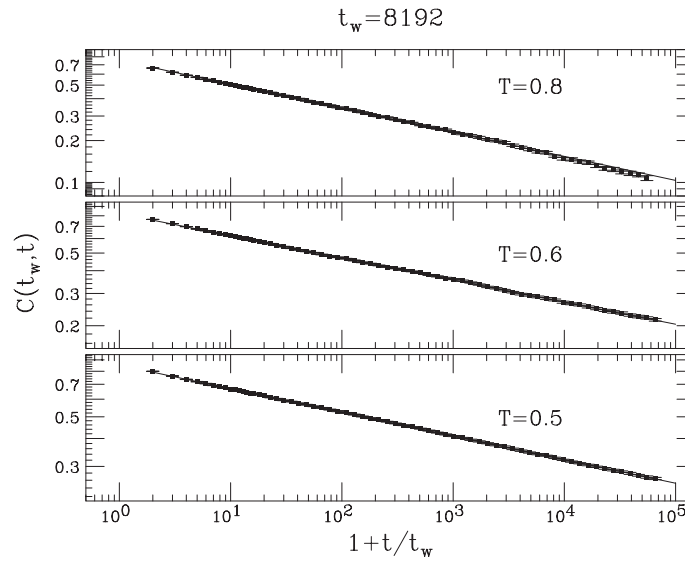
$t_w=8192$

Figure 1. Double logarithmic plot of the correlation function obtained with SUE using a heat-bath algorithm, versus $1 + t/t_w$, for $t_w = 8192$ and $T = 0.5, 0.6$ and 0.8. A single power law fits satisfactorily the data for $t > t_w$ at all three temperatures.

$t_w=9 \times 10^5$
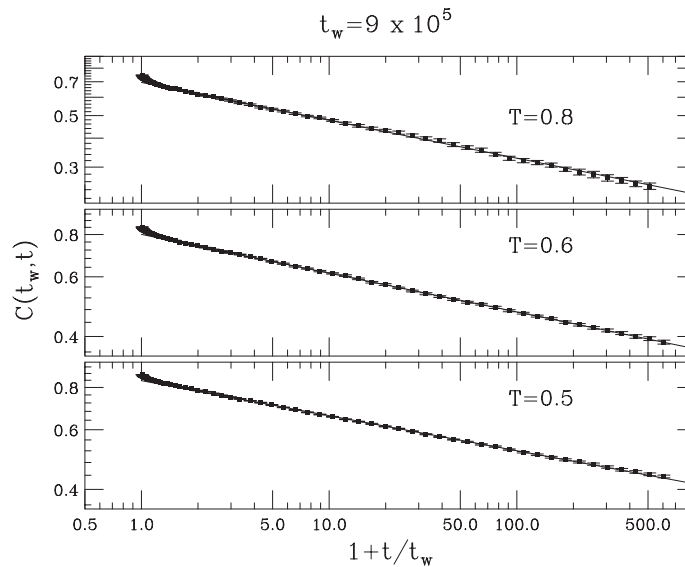
Figure 2. As in figure 2, for $t_w = 884\,736$.

To obtain the coefficients $A(t_w)$ and $\alpha(t_w)$ we have fitted the correlation functions $C(t, t_w)$ to the functional form in equation (9). Yet, although figures 1–3 suggest a pure power-law behaviour, we have found some dependence of $\alpha(t_w)$ on the fitting window, particularly for $T = 0.5$. Specifically, at $T = 0.5\,\alpha(t_w = 8192)$ grows 10% if the fit is performed for $t_w \leqslant t \leqslant 10^5 t_w$ as compared to the fit in the window $t_w \leqslant t \leqslant 10 t_w$. Not taking care of this could be dangerous since, obviously, the larger the t_w shorter the achievable t/t_w.
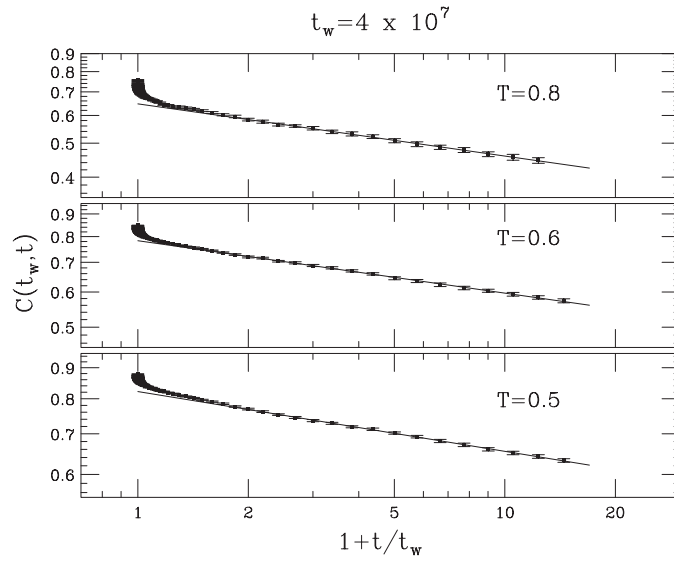
Figure 3. As in figure 2, for $t_w = 39\,813\,120$.

Thus, in order to isolate the t_w-dependence we have restricted ourselves to the fitting range $t_w \leqslant t \leqslant 10t_w$. On the other hand, the prefactor $A(t_w)$ is basically independent of the fitting window. Another tricky point is the error estimate for $A(t_w)$ and $\alpha(t_w)$. It is clear that standard techniques (χ^2 minimization) work poorly for tremendously correlated stochastic variables such as $C(t, t_w)$ for successive t. In order to have a (hopefully) reasonable estimate we have turned to a jack-knife procedure on the fitted coefficients themselves.

Our results for the prefactor $A(t_w)$ and the exponent $\alpha(t_w)$ of the heat-bath dynamics in the $D = 3$ case, can be found in figures 4 ($T = 0.8$), 5 ($T = 0.6$) and 6 ($T = 0.5$). The results for $t_w < 8192$ were obtained on a PC (see section 2), while for $t_w \geqslant 8192$ SUE data were used. Actually, we have found that the functional form (9) is not suitable for very small t_w, thus there is a lower, temperature-dependent, cut-off on the t_w values shown in figures 4–6.

As far as the exponent $\alpha(t_w)$ is concerned (see the upper part of figures 4–6), we have a significant growth for $t_w < 10^6$. For larger t_w, $\alpha(t_w)$ is constant within errors. This (asymptotic?) value would correspond to the inverse of the large waiting-time limit of the Rieger *et al* [9] ageing exponent $\lambda(T, t_w)$. Interestingly enough, the values of $\alpha(t_w)$ for the largest achieved t_w seem to be proportional to T_c/T. On the other hand, the prefactor $A(t_w)$ (see the lower part of figures 4–6) clearly decreases in all the simulated t_w range. A power law seems to be appropriate for this decay, the exponent being precisely the exponent $x(T)$ in equation (3). Thus, we find $x(0.7T_c) = 0.016$, $x(0.53T_c) = 0.0064$, $x(0.44T_c) = 0.0049$.

We have found similar results (see figure 7) for the Metropolis dynamics of the Edwards–Anderson model with binary-coupling distributions in three dimensions ($T = 0.7T_c$) and four dimensions ($T = 0.6T_c$). For all the simulated t_w ($<10^6$) we have found a growing trend in the exponent $\alpha(t_w)$. As usual, the dynamics is faster in four dimensions than in three dimensions (note how the $\alpha(t_w)$ is smaller but more rapidly growing). The prefactor $A(t_w)$ decreases. A power law seems to be adequate for this decay in $D = 3$, but this is less clear for the four-dimensional lattice.
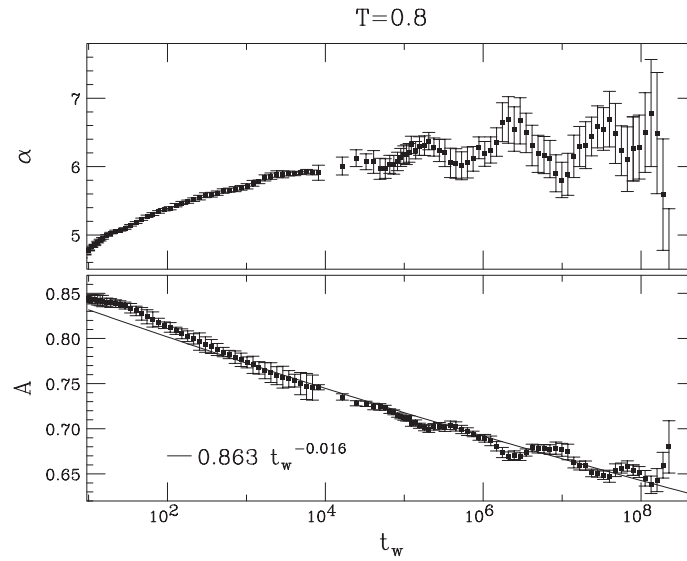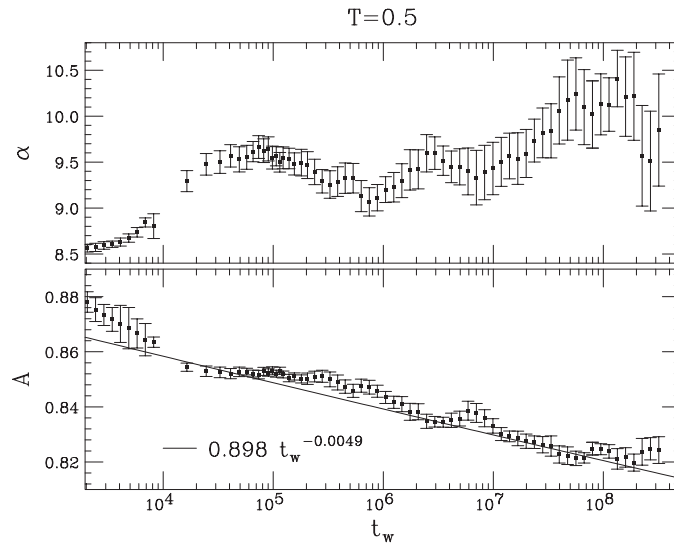
Figure 4. Top: exponent $\alpha(t_w)$ defined in equation (9) versus t_w for the heat-bath dynamics of the Edwards–Anderson model in $D = 3$, at $0.7T_c$. Data for $t_w < 8192$ were obtained on a PC, while for larger t_w correlation functions from SUE were used. Bottom: same as in the top panel for the prefactor $A(t_w)$, defined in equation (9). The solid line was obtained fitting for $t_w \geqslant 8192$.
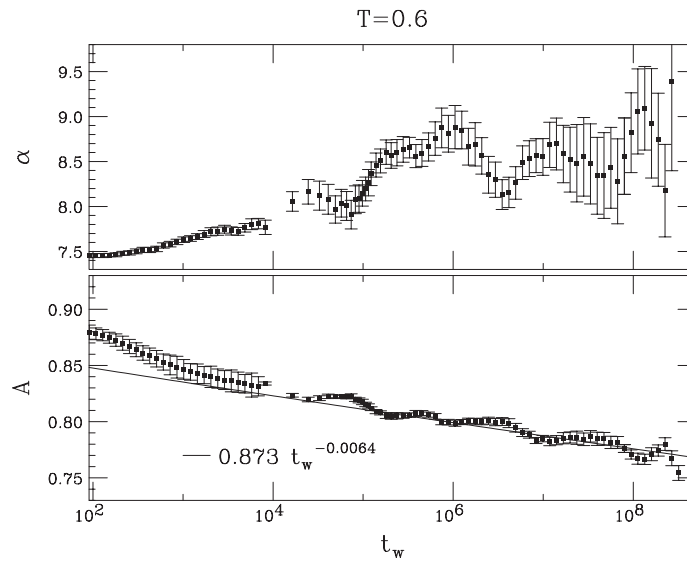

Figure 5. As in figure 4 for $0.53T_c$.

Up to now, we have only confirmed that the dynamics for $t \geqslant t_w$ and $t_w > 10^6$ can be rather well described by the cross-over [20, 9] formula (3), and we have given an explicit form for the cross-over function $\Phi(y)$ (see equation (9)). We are thus predicting that $C(t_w, t_w)$ should vanish in the infinite waiting-time limit: dynamic ultrametricity reduces to a trivial $0 = 0$ statement in the $t \geqslant t_w$ time sector. Therefore, non-trivial statements about dynamical

T=0.5

**Figure 6.** As in figure 4 for $0.44T_c$.
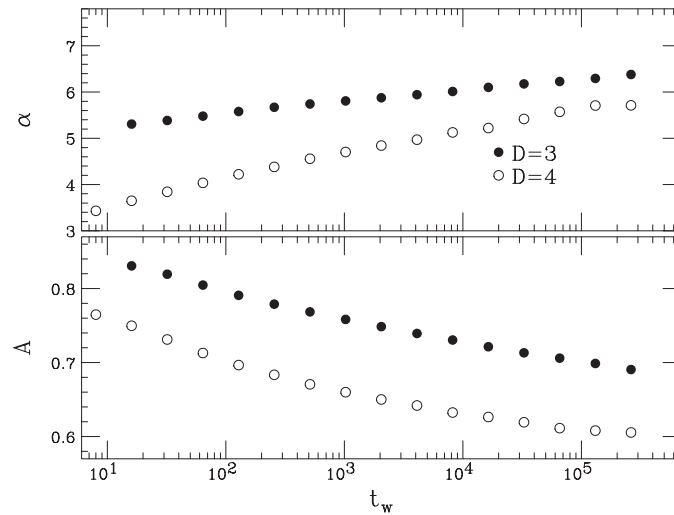
Figure 7. Top: exponent $\alpha(t_w)$ defined in equation (9) versus t_w for the Metropolis dynamics of the Edwards–Anderson model in $D = 3$ at $T = 0.8$ (full circles) and $D = 4$ at $T = 1.2$ (empty circles). Bottom: same as in the top panel for the prefactor $A(t_w)$, defined in equation (9).

ultrametricity should regard times $1 \ll t \ll t_w$. In this time sector, the cross-over formula (3) predicts a time-translational invariant ($\mu = 0$) power-law decay, $C(t, t_w) = \Phi(0)/t^{x(T)}$. This prediction is, of course, non-ultrametric.

However, one could wonder about the presence of more than one time sector. In fact, as one can see in figure 8 for the Metropolis dynamics of the 3D Edwards–Anderson model at $T = 0.7T_c$, small but measurable deviations from time-translation invariance appear at
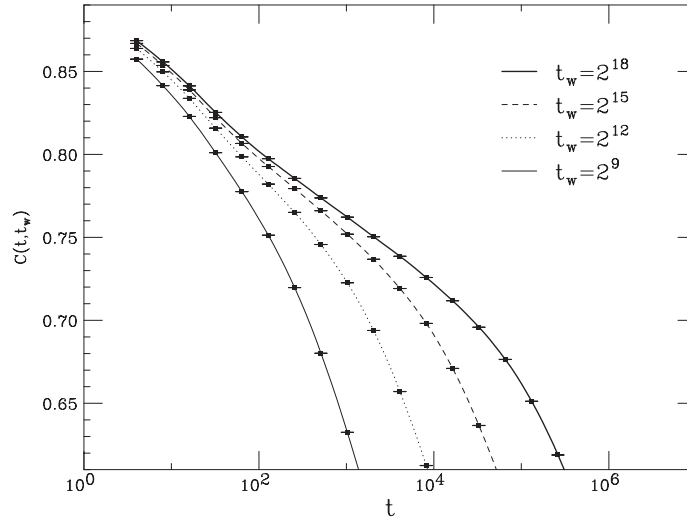
Figure 8. Top: the spin correlation function as a function of t for several t_w, obtained with the Metropolis dynamics of the 3D Edwards–Anderson model at $T = 0.7T_c$ (the lines are mere guides to the eye). Data were obtained in a $L = 33$ lattice simulated on a PC. The plot concentrates on the region of large C in order to discuss the time-translation invariant regime (see text).
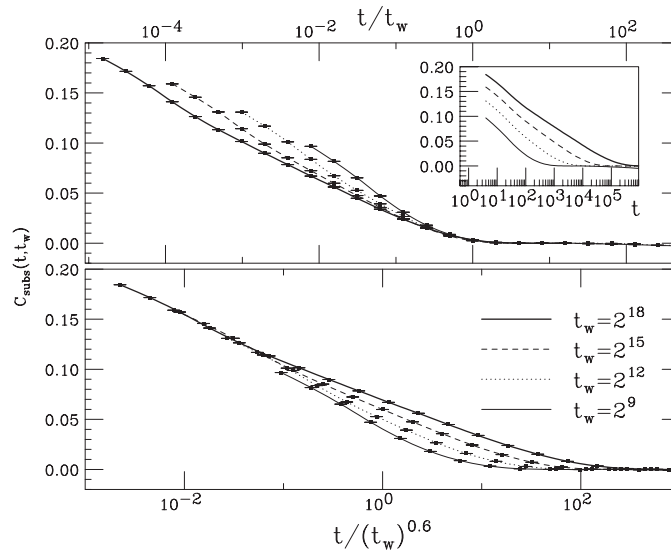
Figure 9. Top: the subtracted correlation function defined in equation (10) as a function of t/t_w, for the Metropolis dynamics of the 3D Edwards–Anderson model at $T = 0.7T_c$ (the lines are mere guides to the eye). The inset shows $C_{\text{subs}}(t, t_w)$ as a function of time, for the different t_w. Bottom: $C_{\text{subs}}(t, t_w)$, from the same simulation as above, versus the (not dimensionless!) ratio $t/t_w^{0.6}$.

$t/t_w \sim 10^{-3}$. In order to explore the regime $1 \ll t \ll t_w$, we have introduced a subtracted correlation function:

$$C_{\text{subs}}(t, t_w) = C(t, t_w) - A(t_w) \left(1 + \frac{t}{t_w}\right)^{1/\alpha(t_w)} \tag{10}$$
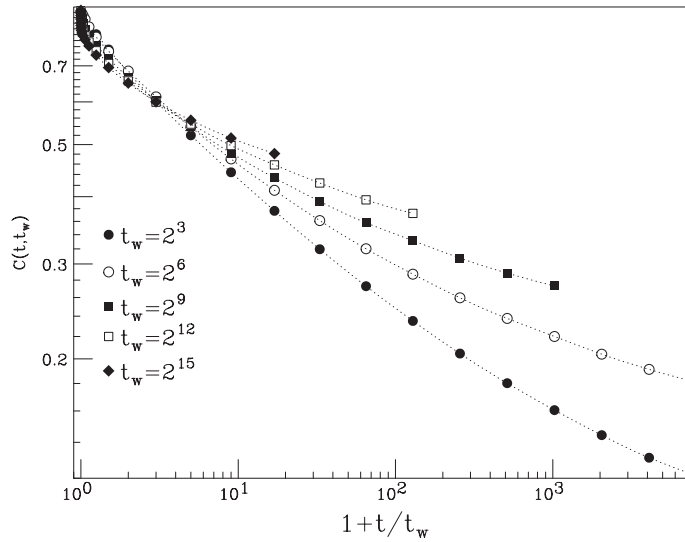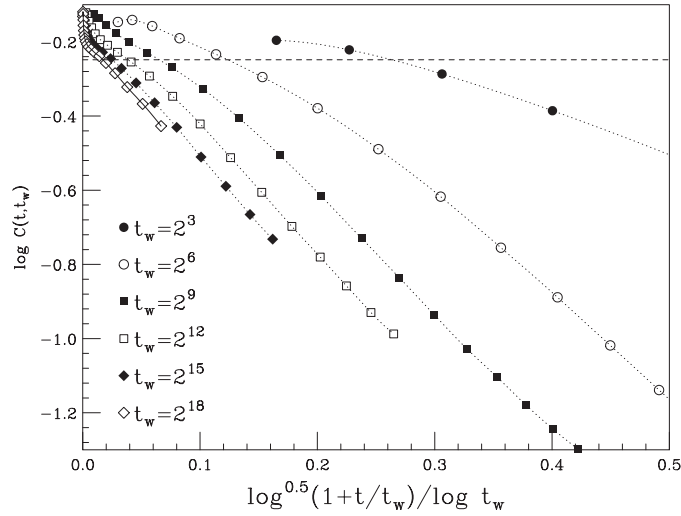
Figure 10. Double logarithmic plot of $C(t, t_w)$ versus $1 + t/t_w$ for the Metropolis dynamics of the infinite-dimensional $z = 6$ Viana–Bray model in ($N = 10^6$, $T = 0.8 = 0.34T_c$). The clearly measurable curvature indicates deviations from equation (9).

where $A(t_w)$ and $\alpha(t_w)$ are, of course, the coefficients obtained in the fit to the functional form (9) for each t_w. To motivate it, let us recall that the quasi-equilibrium regime is realized in the case where t_w goes to infinity at fixed t. Naive scaling predicts that $C(t + t_w, t_w)$ goes to a non-trivial function of t/t_w when t_w goes to infinity. However, we have already remarked that we could also have a sub-ageing contribution and find a non-trivial behaviour in the region where t is of order t_w^μ with $\mu < 1$. In this region we have $1 \ll t \ll t_w$, but we may not be in the quasi-equilibrium regime as far as t goes to infinity and it is not fixed. One reason for studying $C_{\text{subs}}(t, t_w)$ is that in the case of multitime sectors, the correlation function is the sum of contributions from each time sector and the subtraction helps to identify the given time sector.

We show in figure 9 $C_{\text{subs}}(t, t_w)$ for $D = 3$, $T = 0.8$, as a function of t/t_w (upper panel) and as a function of $t/t_w^{0.6}$ (lower panel). It is clear that C_{subs} can be described as a function of t/t_w for $t > 0.1t_w$, but that strong deviations are present for smaller times. The inset in the top panel of figure 9 also shows that $C_{\text{subs}}(t, t_w)$ is *not* a function of t, as the cross-over formula (3) would suggest. On the other hand, the lower part of figure 9 shows that $C_{\text{subs}}(t, t_w)$ seems really a function[9] of $t/t_w^{0.6}$ over two decades (corresponding to the decay from $C_{\text{subs}} = 0.18$ to $C_{\text{subs}} = 0.1$). To our knowledge, this is the first time that a time sector different from $\mu = 0$ (time-translational invariant) and $\mu = 1$ (full ageing) has been studied. A cautionary remark is in order, though, for one cannot exclude that the $t/t_w^{0.6}$ scaling could stop applying at much larger t_w. Unfortunately, because of the small number of simulated samples, we have not been able to repeat this analysis with SUE data: the t/t_w scaling and the $t/t_w^{0.6}$ scaling for C_{subst} look indistinguishable within error bars.

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[9] Note that the numerical value of the quotient $t/t_w^{0.6}$ is not invariant under a change of time units, so that one should really speak about $t/(\tau_0^{0.4} t_w^{0.6})$, τ_0 being the time unit used. Thus, we do not attribute any special significance to $t/t_w^{0.6} = 1$.

Figure 11. Logarithm of $C(t, t_w)$ versus $\sqrt{\log(1 + t/t_w)}/\log t_w$ (see equation (11)) for the Metropolis dynamics of the infinite-dimensional $z = 6$ Viana–Bray model in ($N = 10^6$, $T = 0.8 = 0.34 T_c$). The dashed horizontal line is the logarithm of the $q_{EA} = 0.78$ [26].

It is clear that more work needs to be done in order to design an efficient protocol for studying the different time sectors. Yet, we hope that the reader will be convinced that it should be feasible.

The Metropolis dynamics of the infinite-dimensional $z = 6$ Viana–Bray model is rather different. As we show in figure 10, the long-time decay of $C(t, t_w)$ is *not* a power law. Thus, it cannot be fitted with equation (9). We have found (see figure 11) that a fit to

$$C(t, t_w) = S(t_w) \exp\left[-\eta(t_w)\frac{\sqrt{\log(1 + t/t_w)}}{\log t_w}\right] \qquad t > t_w \qquad (11)$$

is rather adequate. Moreover, $S(t_w)$ and $\eta(t_w)$ seem to have a well-defined limit for large t_w, $S(t_w \to \infty)$ being quite close to the Edwards–Anderson order-parameter $q_{EA} = 0.78$ that in this model has been computed [26] at the one-step level of replica symmetry breaking. If one believes that our results are almost asymptotic, so that the large t_w correlation function truly is

$$C(t, t_w) \approx q_{EA} \exp\left[-\eta\frac{\sqrt{\log(1 + t/t_w)}}{\log t_w}\right] \qquad t > t_w \qquad (12)$$

the rather amazing conclusion is reached that this ultrametric system (from the point of view of statics [3]) is not ultrametric from the point of view of dynamics! It has been pointed out [11] that dynamical ultrametricity implies (under reasonable hypothesis) statical ultrametricity, however, our results suggest that dynamical ultrametricity may not be a necessary condition for the validity of RSB. The reader can check that the weaker property of separation of timescales holds: let $t(C; t_w)$ be the time necessary for the correlation function at waiting time t_w to reach the value C. One has

$$\lim_{t_w \to \infty} \frac{t(C_1; t_w)}{t(C_2; t_w)} = 0 \qquad \text{if} \quad C_1 > C_2. \qquad (13)$$
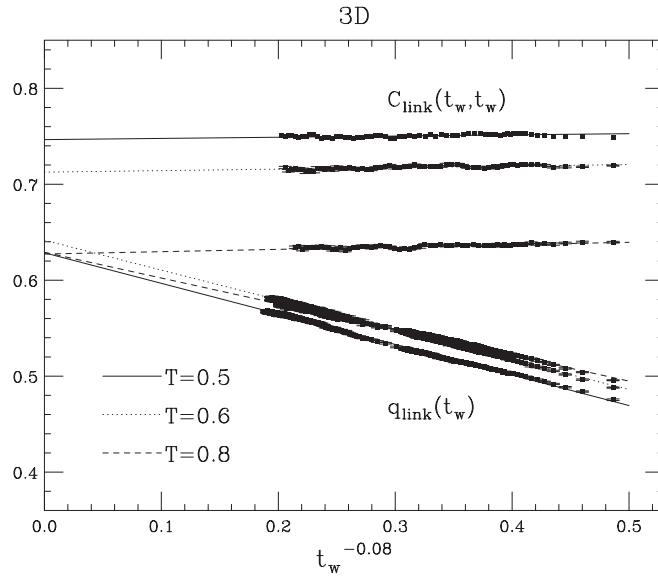
Figure 12. $C_{link}(t_w, t_w)$ and $q_{link}(t_w)$ versus $t_w^{-0.08}$, for the heat-bath dynamics of the 3D Edwards–Anderson model at temperatures $T = 0.7T_c$, $0.53T_c$ and $0.44T_c$. Lines are linear fits.

It is somehow disappointing, though, that this property is also expected to hold for the Langevin dynamics of the disordered ferromagnet [8].

4. The link-overlap

In recent years, a new picture of the low temperature phase of spin-glasses has been put forward [15], the so-called TNT picture. These authors propose that the link-overlap defined in equation (8) should have a trivial distribution function (a Dirac delta) in the thermodynamic limit (for a recent study of the link-overlap and related quantities, see [27]). On the other hand, the spin-overlap defined in equation (7), would have a non-trivial distribution function, as predicted by RSB [3]. In contradiction with RSB, the TNT picture requires that the large waiting-time limit of $C_{link}(t_w, t_w)$ be equal to the limit of $q_{link}(t_w)$ (since there would be only a possible value for this quantity!). This can be checked in our simulations. In figure 12 we show $C_{link}(t_w, t_w)$ and $q_{link}(t_w)$ obtained from SUE, for temperatures $T = 0.7T_c$, $T = 0.53T_c$ and $T = 0.44T_c$, as a function of $t_w^{-0.08}$. The extrapolation as a function of $t_w^{-0.08}$ was suggested by the fact that $q_{link}(t_w)$ is roughly a linear function of this variable. Some arguments for its validity have also been given in previous work [27, 28, 24]. We note that $q_{link}(t_w)$ has a very mild temperature dependence. Furthermore, for large t_w, $q_{link}(t_w, T = 0.6)$ and $q_{link}(t_w, T = 0.5)$ are on top of each other (within error bars), while $q_{link}(t_w, T = 0.8)$ is fast approaching them. Indeed, one would expect $q_{link}(t_w = \infty, T = 0.5) > q_{link}(t_w = \infty, T = 0.6) > q_{link}(t_w = \infty, T = 0.8)$, so that the three lines should cross. The fact that the infinite waiting-time limit of $q_{link}(t_w, T = 0.5)$ is smaller than $q_{link}(t_w = \infty, T = 0.6) > q_{link}(t_w = \infty, T = 0.8)$, indicates that for $T = 0.5$ the data should probably extrapolate like $t_w^{-\eta}$ with $\eta < 0.08$. On the other hand, $C_{link}(t_w, t_w)$ shows a much stronger temperature dependence and, for $T = 0.5$ and $T = 0.6$, is basically t_w independent. It seems plausible that, for $T = 0.8$, the limiting values of $C_{link}(t_w, t_w)$ and $q_{link}(t_w)$ will be fairly close. However, for

$$T=1.2 \ , \ D=4$$
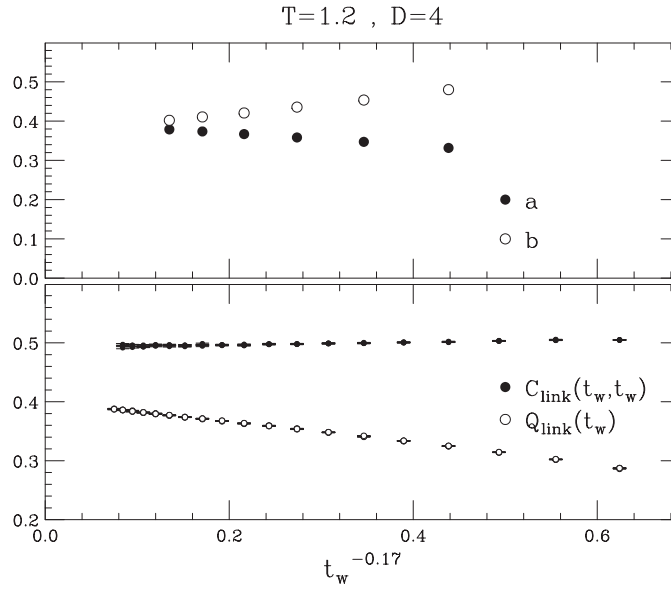
Figure 13. Top: coefficients $a(t_w)$ and $b(t_w)$ (defined in equation (14)), versus $t_w^{-0.17}$, for the Metropolis dynamics of the 4D Edwards–Anderson model at temperatures $T = 0.6T_c$. The fits have been performed in the window $0.1 < C^2(t, t_w) < 0.3$. Bottom: $C_{\text{link}}(t_w, t_w)$ and $q_{\text{link}}(t_w)$ versus $t_w^{-0.17}$, for the Metropolis dynamics of the 4D Edwards–Anderson model at temperatures $T = 0.6T_c$.

$T = 0.5$ and 0.6 the limits will be noticeably different unless the dynamics changes drastically at larger t_w. Interestingly enough, in four dimensions and $T = 0.6T_c$ (see figure 13, lower part), $C_{\text{link}}(t_w, t_w)$ and $q_{\text{link}}(t_w)$ seem to be linear in $t_w^{-0.17}$, and to extrapolate to different values.

Another interesting question is regarding the ageing properties of the link correlation function. From the droplet [14] and TNT pictures of spin-glasses, one would not expect that $C_{\text{link}}(t, t_w)$ would age (at least, for large enough t_w). On the other hand, the RSB picture expects ageing properties akin to the ones of $C(t, t_w)$. To check for this, one could just look to $C_{\text{link}}(t, t_w)$ as a function of $C^2(t, t_w)$ (see figure 14). If one concentrates in a small window of C^2, $0.2 < C^2 < 0.35$ a linear description is perfectly adequate:

$$C_{\text{link}}(t, t_w) = a(t_w) + b(t_w)C^2(t, t_w). \tag{14}$$

The question now translates to the behaviour of $a(t_w)$ and $b(t_w)$. The TNT and droplet pictures predict that $b(t_w)$ tend to zero and $a(t_w)$ tend to the large waiting-time limit of $C_{\text{link}}(t_w, t_w)$. On the other hand, RSB predicts a non-vanishing limit of $b(t_w)$.

In figure 15, we show coefficients $a(t_w)$ (bottom) and $b(t_w)$ (top), as a function of $t_w^{-0.08}$, for the heat-bath dynamics of the 3D Edwards–Anderson model, at $T = 0.5, 0.6$ and 0.8. Obviously, the lower the T, the smaller the range of t_w shown, because $C^2(t, t_w)$ does not reach 0.2 for all the t_w in our simulation time. The errors in $a(t_w)$ and $b(t_w)$ have been calculated with a jack-knife procedure. The coefficients seem to have a nice linear behaviour in $t_w^{-0.08}$. One thus concludes that, unless a drastic change arises in the behaviour of $a(t_w)$ and $b(t_w)$, the slope $b(t_w)$ will not vanish asymptotically. This implies that $C_{\text{link}}(t, t_w)$ should age as $C^2(t, t_w)$ does. One must acknowledge, however, that the conclusion is less sound for $T = 0.8$ than for the lower temperatures. In fact, for $T = 0.8$ the infinite-volume extrapolation
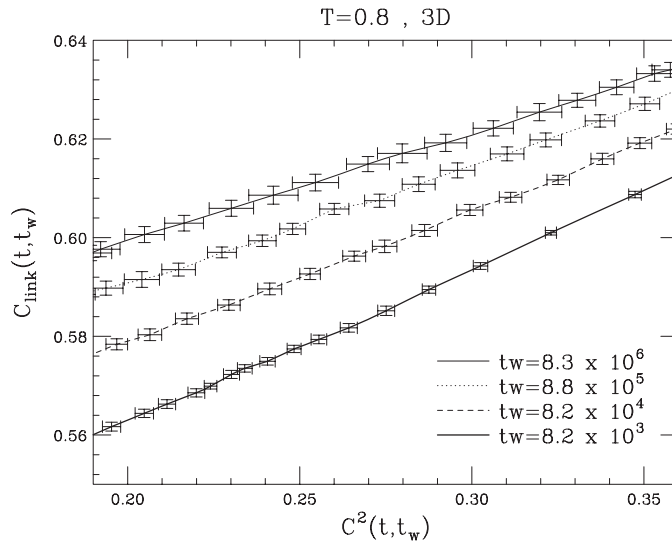
Figure 14. $C_{\text{link}}(t, t_w)$ versus $C^2(t, t_w)$, obtained in SUE for the heat-bath dynamics of the 3D Edwards–Anderson model at $T = 0.7T_c$. The plot is restricted to the window of C^2 where a linear dependence is found for all the studied t_w. The lines are linear fits.
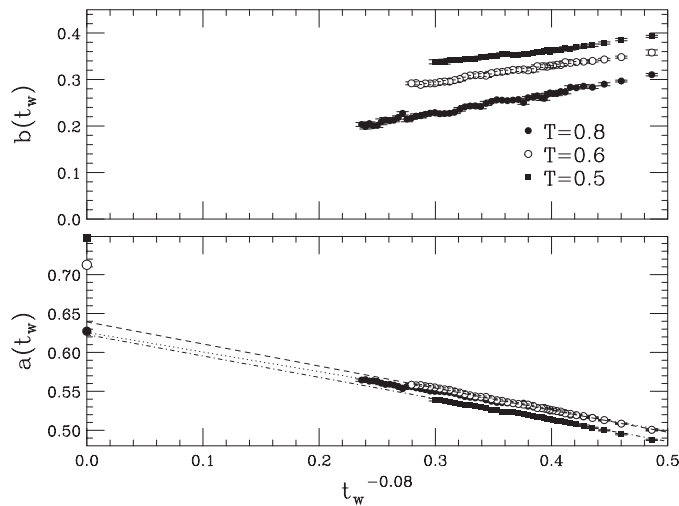
Figure 15. Top: coefficient $b(t_w)$, defined in equation (14), obtained in SUE for the heat-bath dynamics of the 3D Edwards–Anderson model versus $t_w^{-0.08}$, for temperatures $T = 0.7T_c$, $0.53T_c$ and $0.44T_c$. Bottom: as in top part, for coefficient $a(t_w)$, defined in equation (14). The lines are linear fits. We also plot in the vertical axis the infinite-limit extrapolation for $C_{\text{link}}(t_w, t_w)$ (see figure 12) for all three temperatures. Note that for $T = 0.8$ the infinite t_w limits of $a(t_w)$ and of $C_{\text{link}}(t_w, t_w)$ seem compatible.

of $a(t_w)$ (see bottom part of figure 15) is quite close to the large time-limit of $C_{\text{link}}(t_w, t_w)$, as TNT would predict. For the lower temperatures these extrapolations are clearly different. The data in four dimensions also support the RSB prediction (see the top part of figure 13).

5. Conclusions

In this work, we report the results of a large scale Monte Carlo simulation of the heat bath dynamics of the three-dimensional Edwards–Anderson model with binary coupling at temperatures $T = 0.7T_c$, $0.53T_c$ and $0.44T_c$ (see table 1). The long times achieved and the large lattices studied (60^3) have been made possible by the SUE machine. In addition, shorter but more precise[10] simulations are presented for the Metropolis dynamics of the same model in 3D, 4D and the infinite-dimensional Viana–Bray model.

For the spin correlation function in $D = 3$ and $D = 4$, we find that a slightly modified power law (see equation (9)), well describes the data for $t > t_w$. This formula is identical to the cross-over like parametrization (3) proposed by Rieger *et al* [20, 9]. However, in the regime $1 \ll t \ll t_w$ a different behaviour is observed. A numerical procedure is proposed for the study of the different time sectors believed to exist in spin-glasses dynamics [8]. Indeed, a time sector with characteristic exponent $\mu = 0.6$ is observed for the first time, we believe, although to firmly establish this result will require very precise simulations at still larger t_w. In spite of this, the very slow t_w evolution of $A(t_w)$ and $\alpha(t_w)$ (see equation (9)), makes us believe that in thermoremanent magnetization measurements (usually restricted to the range $10^{-2}t_w < t < 10^2 t_w$), a perfect full-ageing behaviour occurs, in agreement with a recent experiment [19], and mild disagreement with older ones [6]. Although it is somehow disappointing that the study of the (in)existence of more than two time sectors in spin-glass dynamics should be restricted to simulations, we still think that it should be feasible. Nevertheless, one can always hope that new experimental techniques and protocols will eventually be able to explore this time regime.

In infinite dimensions, or at least for the $z = 6$ Viana–Bray model, we have found a different scaling. The decay of the spin correlation function is not a power law. Moreover, the limiting functional-form equation (12) is *not* dynamically ultrametric. Although clearly more work is needed to establish this result, it suggests that dynamical ultrametricity may not be such an interesting property as previously thought.

We have also considered the ageing properties of the link-overlap and the link correlation function, both in three dimensions and in four dimensions. We have concluded that, unless a drastic change in the dynamical properties arises for t_w larger than studied here, the link correlation function should age precisely in the same way as the spin correlation function. This is in plain disagreement with the droplet and the TNT pictures of the spin-glass phase. However, one must acknowledge that the data at the highest studied temperature in three dimensions ($T = 0.7T_c$) are not incompatible with the TNT picture.

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